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Geometric Estimation of Fixed Points of Lipschitzian Mappings, II

T. E. WILLIAMSON, JR.

Montclair State College, Upper Montclair, N.J. 07043

Submitted by C. L. Dolph

A mapping T of a subset D of a Hilbert space H into H is Lipschitzian with constant $L > 0$ provided that for every $x, y \in D$

$$\| Tx - Ty \| \leqslant L \| x - y \| . \tag{1}$$

In [11] the following is proved:

THEOREM 1. *Let H be a Hilbert space, $T: D \subseteq H \to H$ a Lipschitzian mapping of D into H with constant $L > 0$. Let $x \in D$, and suppose $x \neq Tx$.*

(a) *If $L < 1$, then $F(T)$ is contained in the closed ball $B(c, r)$ centered at $c \in H$ with radius r, where*

$$c = [1 - (1 - L^2)^{-1}]\, x + (1 - L^2)^{-1}\, Tx$$

and

$$r = \| x - Tx \| \cdot L \cdot [1 - L^2]^{-1}. \tag{2}$$

(b) *If $L > 1$, then $F(T)$ is contained in the complement of the open ball interior $B(c, r)$, where c and r are given by the equations in part* (a).

(c) *If $L = 1$, then $F(T)$ is contained in the closed half space containing Tx and bounded by the hyperplane* (*codimension* $= 1$) *which perpendicularly bisects the line segment* $[x, Tx]$.

In this paper we investigate (for $L \leqslant 1$) the construction of sequences to approximate a fixed point with discussion of error estimation. For $L \leqslant 1$ and $H = \mathbb{R}^1$ a rapidly convergent scheme is given, and for $L < 1$ and general H a scheme is given which appears to be especially effective computationally for values of L near to 1.

For notation, B.P.C. refers to the strict contraction principle of Banach–Picard–Caccioppoli, R_x refers to the set given in Theorem 1 in which the fixed point set $F(T)$ must be contained, and for $L \neq 1$ let

$$\lambda = (1 - L^2)^{-1}. \tag{3}$$

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1. $L < 1$

Remark 1. Theorem 1a could be used in conjunction with the iteration scheme $\{x_n = T^n(x_0)\}$ to improve the estimate at some final Nth step. This situation could occur in the case $T(D) \not\subseteq D$ and at some N, $T^N(x_0) \notin D$, which terminates the construction, or if a predetermined number N of iterates were to be computed. Let $t = \| Tx_0 - x_0 \|$. At the Nth step the B.P.C. estimate is

$$\| p - T^N(x_0)\| \leqslant tL^N(1 - L)^{-1}, \qquad \text{where} \qquad p \in F(T).$$

However, by choosing

$$c = (1 - \lambda)\, T^{N-1}(x_0) + \lambda T^N(x_0) \tag{4}$$

we have

$$\| p - c \| \leqslant \| T^N(x_0) - T^{N-1}(x_0)\| \cdot L \cdot \lambda \leqslant t \cdot L^N \cdot \lambda,$$

which is an improvement by a factor $(1 + L)^{-1}$.

The maximality of the set R_x and the gain over the B.P.C. ball (see [11]) invite us to attempt to construct a sequence of approximants, using Theorem 1a at each step, with the hope of reflecting this gain in producing an error estimate superior to B.P.C. at the corresponding step. The natural choice is to try choosing (consecutively) the center of the approximating ball (given at the previous step by Theorem 1a) as the point from which to iterate for the next step. More precisely, the scheme $\{c_n\}$ where $c_0 = x_0$ and for $n \geqslant 1$,

$$c_n = (1 - \lambda)\, c_{n-1} + \lambda T(c_{n-1}).$$

Unfortunately this extrapolation scheme need not even converge. This can be seen by choosing $D = H = \mathbb{R}^1$, $\frac{1}{2} < L < 1$, $Tx = -Lx$ for all $x \in \mathbb{R}$. Then for any $x_0 \neq 0$, the sequence $\{c_n\}$ diverges, i.e., $\| c_n \| \to \infty$ as $n \to \infty$. Although the sequence diverges, it is noteworthy that regardless of the choices of x_0 and L, for this example, it is always true that $R_{c_0} \cap R_{c_1} = \{0\}$. That is, the fixed point is found in just two iteration steps. This suggests that closer investigation of (at least) pairwise intersections of the approximating sets may be fruitful.

For the following discussion, assume $D = H$ because of the (possibly) extrapolated nature of the method.

We note that the point z_1 on the segment $[x, Tx]$ which is the closest point of R_x to x, divides the segment $[x, Tx]$ in the ratio L: 1, that is, $\| Tx - z_1 \| = L \| x - z_1 \|$.

THEOREM 2. *Let $D = H$ and $T: D \to H$ be Lipschitz of constant $L < 1$. Choose $d_0 \in D$. Let $d_1 = \lambda T d_0 + (1 - \lambda)\, d_0$ and $r_1 = \| d_0 - Td_0 \| L\lambda$. $F(T) \subseteq B(d_1, r_1)$. If d_{n-1} and r_{n-1}, $n > 1$, have been constructed, define*

$$t_n = \| Td_{n-1} - d_{n-1} \|.$$

If $t_n = 0$, d_{n-1} *is a fixed point.*
If $t_n > (1 + L)\, r_{n-1}$, $F(T) = \varnothing$.
If $0 < t_n \leqslant (1 - L^2)(1 + L)^{-1/2}\, r_{n-1}$, *then let*

$$r_n = t_n L \cdot \lambda, \qquad \textit{and} \qquad d_n = \lambda T\, d_{n-1} + (1 - \lambda)\, d_{n-1}\,. \tag{5}$$

If $(1 - L^2)(1 + L^2)^{-1/2}\, r_{n-1} \leqslant t_n \leqslant (1 + L)\, r_{n-1}$, *then let*

$$r_n = \left[\frac{1}{2}(1 + L^2)(r_{n-1})^2 - \frac{(t_n)^2}{4} - \frac{(1 - L^2)^2}{4(t_n)^2}(r_{n-1})^4\right]^{1/2} \tag{6}$$

and

$$d_n = \gamma_n T\, d_{n-1} + (1 - \gamma_n)\, d_{n-1}\,, \tag{7}$$

where

$$\gamma_n = (t_n)^{-1}\,[(r_{n-1})^2 - (r_n)^2]^{1/2}. \tag{8}$$

Then $F(T) \subseteq B(d_n\,, r_n)$ *and* $r_n \leqslant L r_{n-1} \leqslant L^n \lambda t_1$.

Proof. $F(T) \subseteq B(d_1\,, r_1)$ by Theorem 1a. Since the method is a one-step method, it suffices to verify the claims for $n = 2$. $F(T) \subseteq B(d_1\,, r_1)$ and the comment preceding this theorem imply that if $t_2 > (1 + L)\, r_1$, then $F(T) = \varnothing$. For the remainder of the proof we will suppress the subscript and refer to t instead of t_2. As t increases (from zero) the estimating ball $B(c, r) = R_{d_1}$ given by Theorem 1a is taken to be $B(d_2\,, r_2)$ until t becomes large enough that the intersection $B(d_1\,, r_1) \cap B(c, r)$ can be enclosed in a ball of radius smaller than r. This changeover occurs when no pair of diametrical points on the boundary of $B(c, r)$ is contained in $B(d_1\,, r_1)$. (See Fig. 1-II.) So it suffices to determine the value of t at this changeover and to determine r_2 and d_2 for values of t greater than this changeover value, which we will call t_{II}. From Fig. 1-II,

$$(r_1)^2 = (r_2)^2 + \| d_1 - c \|^2 = (r_2)^2 + [r_2 + (L + 1)^{-1}\, t]^2, \tag{9}$$

with $r_2 = r = \lambda \cdot t \cdot L$. Solving for t yields

$$t_{\mathrm{II}} = (1 - L^2)(1 + L^2)^{-1/2}\, r_1\,. \tag{10}$$

For $t \geqslant t_{\mathrm{II}}$, clearly $d_2 = \gamma_2 T\, d_1 + (1 - \gamma_2)\, d_1$, and $\gamma_2 > 0$. Also from Fig. 1-III,

$$\| d_1 - d_2 \|^2 = (r_1)^2 - (r_2)^2. \tag{11}$$

Since

$$d_2 - d_1 = \gamma_2(T d_1 - d_1), \tag{7}$$

then

$$\gamma_2 = t^{-1}[(r_1)^2 - (r_2)^2]^{1/2}. \tag{8}$$

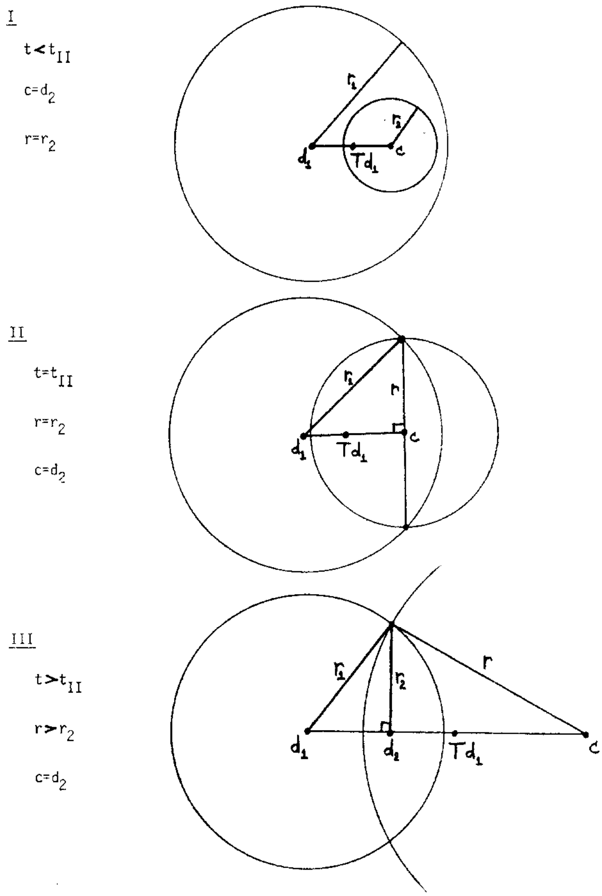

FIGURE 1

To determine r_2 for $t \geqslant t_{\mathrm{II}}$ we see from Fig. 1-III and Theorem 1a that r_2 must satisfy:

$$(r_1)^2 = (r_2)^2 + \| d_1 - d_2 \|^2, \tag{11}$$

$$(r)^2 = (r_2)^2 + \| d_2 - c \|^2, \tag{12}$$

$$\| d_1 - d_2 \| + \| d_2 - c \| = \| d_1 - c \| , \tag{13}$$

$$r = L\lambda t, \tag{14}$$

$$\| d_1 - c \| = r + (L + 1)^{-1} t. \tag{15}$$

Solving these equations gives r_2 as claimed.

Hence $F(T) \subseteq B(d_1, r_1) \cap B(c, r) \subseteq B(d_2, r_2)$.

For the error estimate, it suffices to show $r_2 \leqslant Lr_1$. Since r_2 increases linearly for $t \leqslant t_{\mathrm{II}}$, it is clear that the maximum value of r_2 must occur at some value of $t\,(=t_{\max}) \geqslant t_{\mathrm{II}}$. For $t \geqslant t_{\mathrm{II}}$, taking dr_2/dt, we find $t_{\max} = \lambda^{-1/2} r_1$, and at $t_{\max}$,

$$r_2 = Lr_1. \quad \blacksquare$$

Remark 2. $\{r_n\}$ is monotone decreasing, but, in general, the sequence $\{\| d_n - p \|\}$ is not monotone. Although the estimates of Theorem 2 are no better than those of Remark 1, the method here does explain the fast convergence of the example preceding the theorem. Some numerical testing (with $0.9 \leqslant L \leqslant 0.999999$) in the plane has indicated that frequently the geometric method of Theorem 2, when compared to B.P.C., produces error estimates which are better by a factor 10^{15} within 100 iteration steps. Some examples were found for which the B.P.C. method converged faster than the geometric method, but this appeared to happen only when both methods were converging rapidly.

The restriction $D = H$ is not essential. All that we need is that $B(d_1, r_1) \subseteq D$ or that D be a closed ball. We simply "reset" or retract towards d_1, if some d_n fails to lie in $B(d_1, r_1)$.

THEOREM 3. *Let $T: D \subseteq H \to H$ be as in Theorem 2, except that D is not necessarily all of H. Choose $d_0 \in D$ and suppose that $B(d_1, r_1) \subseteq D$, where d_1, r_1 are defined as in Theorem 2. For $n \geqslant 2$ if $\| d_n - d_1 \| \leqslant r_1$ define d_{n+1} as in Theorem 2. If $r_1 < \| d_n - d_1 \| \leqslant r_1 + r_n$, define d_n' and $r_n' \geqslant 0$ by*

$$d_n' = \beta d_1 + (1 - \beta)\, d_n, \tag{16}$$

$$(r_n')^2 = (r_1)^2 - (1 - \beta)^2 \| d_1 - d_n \|^2, \tag{17}$$

where

$$\beta = 2^{-1} - (r_1^2 - r_n^2) \cdot (2 \| d_1 - d_n \|^2)^{-1}. \tag{18}$$

Then $d_n' \in B(d_1, r_1)$ and $r_n' \leqslant r_n$. Define d_{n+1} and r_{n+1} as in Theorem 2 using d_n' and r_n' instead of d_n and r_n. The estimates still hold.

Proof. Note that if $\| d_1 - d_n \| > r_1 + r_n$, then $F(T) = \varnothing$. If $r_1 < \| d_1 - d_n \| \leqslant r_1 + r_n$, d_n' and r_n' are constructed as the center and radius of the smallest ball containing the intersection of $B(d_1, r_1)$ and $B(d_n, r_n)$. The claims follow easily from solving the following three equations:

$$\| d_1 - d_n' \| + \| d_n' - d_n \| = \| d_1 - d_n \|, \tag{19}$$

$$r_1^2 = (r_n')^2 + \| d_1 - d_n' \|^2, \tag{20}$$

$$r_n^2 = (r_n')^2 + \| d_n - d_n' \|^2. \quad \blacksquare \tag{21}$$

In the special case that $D = H =$ Reals, the sets R_x provide a "midpoint" iteration process with very fast convergence.

THEOREM 4. *Let $D = H = \mathbb{R}^1$, and $T: D \to H$ Lipschitzian with $L < 1$. Let $d_0 \in D$. By Theorem 1a, form R_{d_0}, which is an interval $[a_0, b_0]$ with midpoint*

$d_1\ (= \lambda T d_0 + (1-\lambda) d_0)$ *and radius* $r_1\ (= \lambda L \| Td_0 - d_0 \|)$. *For* $n \geqslant 1$, *with* $[a_{n-1}, b_{n-1}]$, d_n, *and* r_n *constructed, define* $[a_n, b_n] = [a_{n-1}, b_{n-1}] \cap R_{d_n}$, $d_{n+1} = \frac{1}{2}(a_n + b_n)$, $r_{n+1} = \frac{1}{2}(b_n - a_n)$. *Then* $F(T) \subseteq [a_n, b_n]$ *and* $r_n \leqslant (L/(1+L))^{n-1} r_1$.

Proof. It suffices to show that $F(T) \subseteq [a_1, b_1]$ and $r_2 \leqslant (L/(1+L))\, r_1$. By definition $[a_1, b_1] = [a_0, b_0] \cap R_{d_1}$. By Theorem 1a, $F(T) \subseteq [a_0, b_0]$ and $F(T) \subseteq R_{d_1}$, hence the first claim holds true. Let $t = \| Td_1 - d_1 \|$ and $R_{d_1} = B(c, r)$, where c and r are given by Theorem 1a for the test point d_1. As t increases $R_{d_1} \subseteq [a_0, b_0]$, until $t\ (= t_{\mathrm{I}})$ satisfies $r_1 = (L+1)^{-1} t + 2r$, which implies $t_{\mathrm{I}} = (1-L)\, r_1$. In other words as t increases define $r_2 = r\ (= L\lambda t)$ until the value t_{I}, beyond which $R_{d_1} \not\subseteq [a_0, b_0]$. For $t_{\mathrm{I}} < t \leqslant (1+L)\, r_1$, the interval $[a_1, b_1]$ has length

$$r_1 - (L+1)^{-1} t = 2r_2, \qquad \text{defining } r_2.$$

Once again, $t > (1+L)\, r_1$ implies $F(T) = \varnothing$. So, as t increases r_2 increases linearly until $t = t_{\mathrm{I}}$, and decreases linearly from $t = t_{\mathrm{I}}$ to $t = (1+L)\, r_1$. So the maximum value possible for r_2 would occur if $t = t_{\mathrm{I}}$, at which point the value of $r_2 = (L/(1+L))\, r_1$.

Remark 3. Theorem 4 provides a convergence rate of $(L/(L+1))^n$ as opposed to the rate L^n of B.P.C. estimates, hence the rate of convergence is always better than $(\frac{1}{2})^n$ for any $L < 1$. $D = H$ is not essential, any convex domain (interval) is sufficient. The existence of a fixed point is not required by the method of Theorem 4, in fact, if for some n $[a_n, b_n] \cap R_{d_{n+1}} = \varnothing$ then $F(T) = \varnothing$. The hypotheses of Theorems 2, 3 and 4 each imply $F(T) \neq \varnothing$

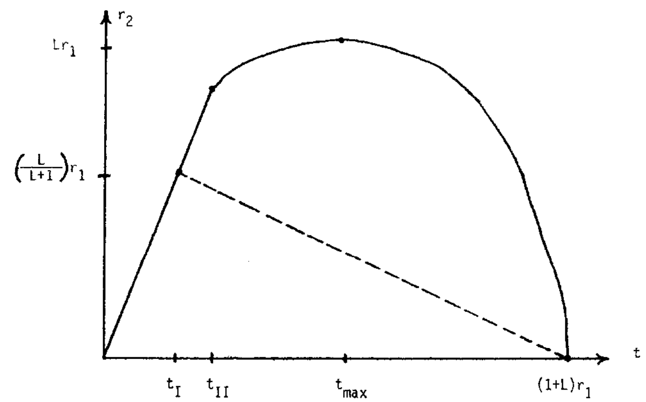

FIG. 2. $t = \| Td_1 - d_1 \|$; $t_{\mathrm{I}} = (1-L)\, r_1$; $t_{\mathrm{II}} = (1-L^2) \cdot (1+L^2)^{-1/2}\, r_1$; $t_{\max} = (1-L^2)^{-1/2}\, r_1$. Solid line—Theorem 2; Broken line—Theorem 4.

(see [11, Theorem 4]). These three theorems are valid for quasi-Lipschitzian mappings, in which case, the existence of fixed points must be assured by other means. See Fig. 2 for a graph of r_2 versus t.

2. $L = 1$

For T nonexpansive ($L = 1$), many results on locating of fixed points are known in a variety of Hilbert and Banach space settings. See, for instance, Krasnoselskiĭ [7], Schaefer [10], Edelstein [4], Browder and Petryshyn [3], Kaniel [6], Petryshyn and Williamson [9]. In general, these have provided various iterative schemes which converge (in norm or weak topology) to a fixed point. Some of the schemes guarantee that successive iterates are getting closer to a fixed point, but none provide error estimates. Our next theorem provides a constructive scheme with error estimates in the case $H = \mathbb{R}^1$. This theorem can be viewed as the "limit" as $L \to 1^-$ of Theorem 4.

THEOREM 5. *Let $H = \mathbb{R}^1$, $D = [a_0, b_0]$ and $T: D \to H$ be Lipschitz of constant $L = 1$. Let d_1 be the midpoint of $[a_0, b_0]$ and $r_1 = \frac{1}{2}(b_0 - a_0)$. For $n \geqslant 1$, if $[a_{n-1}, b_{n-1}]$, d_n, r_n have been constructed and $Td_n \neq d_n$, define $[a_n, b_n] = [a_{n-1}, b_{n-1}] \cap R_{d_n}$, $d_{n+1} = (a_n + b_n)\frac{1}{2}$, $r_{n+1} = \frac{1}{2}(b_n - a_n)$. Then $F(T) \subseteq [a_n, b_n]$ and $r_{n+1} < (\frac{1}{2})^n r_1$. If at any step*

$$[a_{n-1}, b_{n-1}] \cap R_{d_n} = \varnothing, \qquad \textit{then} \qquad F(T) = \varnothing.$$

Proof. It suffices to show $F(T) \subseteq [a_1, b_1]$ and $r_2 < \frac{1}{2}r_1$, provided that $[a_0, b_0] \cap R_{d_1} \neq \varnothing$ and $Td_1 \neq d_1$. By Theorem 1c $F(T) \subseteq R_{d_1}$ which implies $F(T) \subseteq [a_1, b_1]$. To verify the estimate, consider the construction of $[a_1, b_1]$. If $Td_1 > d_1$, then $a_1 = \frac{1}{2}(Td_1 + d_1)$ and $b_1 = b_0$. If $Td_1 < d_1$, then $a_1 = a_0$ and $b_1 = \frac{1}{2}(Td_1 + d_1)$. In either case, $(b_1 - a_1) < \frac{1}{2}(b_0 - a_0)$, that is, $r_2 < \frac{1}{2}r_1$. ∎

Remark 4. For $D = H = \mathbb{R}^1$ and $L = 1$. If two points w_1, w_2 are found such that $(w_1 - Tw_1)(w_2 - Tw_2) < 0$, then either we have $R_{w_1} \cap R_{w_2} = \varnothing$ in which case $F(T) = \varnothing$, or we have produced a bounded interval $[a_0, b_0] = R_{w_1} \cap R_{w_2}$ to which Theorem 5 can be applied. Similar provident points bring other domains into form for the application of Theorem 5.

In higher-dimension Hilbert spaces, we could not utilize Theorem 1c to provide norm error estimates which converge to zero (in norm). However, using slightly differing methods, we could provide "improving" norm error estimates (Theorem 6).

THEOREM 6. *Let $T: B \to B$ be a Lipschitzian mapping with $L \leqslant 1$, where $B = B(q, s)$ is a closed ball in the Hilbert space H. Let $x_0 = q$ and define for $n \geqslant 1$*

$$x_n = (Tx_{n-1} + x_{n-1})\,\tfrac{1}{2} \equiv T^n_{1/2}x_0 . \tag{22}$$

Then for every $p \in F(T)$, and $n \geqslant 0$,

$$\| p - x_n \|^2 \leqslant s^2 - \tfrac{1}{4} \sum_{i=0}^{n-1} t_i^2, \tag{23}$$

where

$$t_i = \| x_i - Tx_i \| , \qquad i \geqslant 0.$$

Proof. Note $F(T) \neq \varnothing$ (Browder [1]), and the convexity of B implies $\{x_n\}$ is defined. Clearly for $n = 0$, $\| p - x_0 \|^2 \leqslant s^2$. If $x_0 \neq Tx_0$, we see that the intersection of B and R_{x_0} is contained in a ball centered at x_1 with radius h_1, where $h_1^2 + (t_0/2)^2 = s^2$. So $F(T) \subseteq B(x_1, h_1)$. If $x_1 \neq Tx_1$, the intersection of $B(x_1, h_1)$ and R_{x_1} is contained in the closed ball centered at x_2 with radius h_2, where $(h_2)^2 + (t_1/2)^2 = (h_1)^2$. Hence $(h_2)^2 = s^2 - \frac{1}{4}(t_0^2 + t_1^2)$. Similarly for $n > 2$. ∎

Remark 5. $\frac{1}{4}\sum_{i=0}^{\infty} t_i^2 \leqslant \| p - x_0 \|^2 \leqslant s^2$ since x_0 is a "reasonable wanderer" (Browder–Petryshyn [3]). If $x_n \neq Tx_n$ the $(n + 1)$st estimate is an improvement over the nth estimate. The sequence $\{x_n\}$ is asymptotically regular, that is, $\| x_n - x_{n-1} \| \to 0$ (Browder–Petryshyn [3]) and $\| x_n - p \|$ is a nonincreasing function of n, for each fixed point p. Without some additional conditions on T we cannot conclude $x_n \to p$, in view of Lindenstrauss' Hilbert space example [5]. Hence, our error estimates, in general, do not converge to zero.

The condition $T(B) \subseteq B$ can be removed by resetting or retracting as in Theorem 3, with no increase in the error estimate. Petryshyn [8] and Browder–Petryshyn [3] discuss the convergence of sequences formed in a similar manner.

Remark 6. In a general sense, these geometric methods capitalize on large steps ($\| x - Tx \|$ large) for $L = 1$ (Theorems 5 and 6), and on both large and small steps for $L < 1$ (Theorems 2 and 4, see Fig. 2). This latter case is to be compared to B.P.C. estimates ($L < 1$) which only capitalize on small steps.

THEOREM 7. *Let $T: D \subseteq \mathbb{R}^m \to \mathbb{R}^m$ be an isometry. In at most m steps either a point of $F(T)$ is located or $F(T)$ is verified as empty.*

Proof. Let $x_1 \in D$ and suppose $Tx_1 \neq x_1$. Since both T and T^{-1} are Lipschitz of constant $L = 1$, then the set R_{x_1} (for T) and R_{Tx_1} (for T^{-1}) intersect in a hyperplane W_{x_1} of dimension $m - 1$. Since $F(T) = F(T^{-1})$, then $F(T) \subseteq$

$D \cap W_{x_1}$. Continuing, choose $x_2 \in D \cap W_{x_1}$ (if $D \cap W_{x_1}$ is empty, the so is $F(T)$). If $x_2 \neq Tx_2$, then

$$F(T) \subseteq D \cap W_{x_1} \cap W_{x_2},$$

and $W_{x_1} \cap W_{x_2}$ is a hyperplane of dimension $m - 2$, since $x_2 \notin W_{x_2}$. If this process continues for m-steps, then

$$F(T) \subseteq D \cap \left(\bigcap_{i=1}^{m} W_{x_i} \right) \tag{24}$$

and

$$\bigcap_{i=1}^{m} W_{x_i} \text{ consists of a single point.}$$

Either this point is a fixed point or $F(T)$ is empty. ∎

Remark 7. If T is Lipschitz of constant $L = 1$, the set R_x can be characterized as the set of y such that

$$2(Tx - x, y) \geqslant \| Tx \|^2 - \| x \|^2. \tag{25}$$

Hence, if T is an isometry, the inequality becomes an equality for the set W_x. So the above theorem can be viewed as constructing (at most) m linear equations in m unknowns, where the unknowns are the coefficients of y in some basis for $\mathbb{R}^m$.

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